Differentiation by the Chain Rule - Classwork

Suppose you were asked to take the derivatives of the following. Could you do so?

a) $f(x)=(2x+5)^2$ b) $f(x)=(2x+5)^3$ c. $f(x)=(2x+5)^6$ d) $f(x)=\sqrt{2x+5}$

a) causes no problem. b) is also not a problem but multiplying it out so you can take the derivative is a bit of a pain. You are capable of doing c) but clearly do not wish to. But d) can't be done with the knowledge you have.

We now introduce a method of taking derivatives of more complicated expressions. It is called the **chain rule**. If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentable function of x, then $y=f(g(x))$ is a differentable function of x and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ or equivalently, $\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot g'(x)$.

Example 1) If $f(x)=(2x+5)^2$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule b) with chain rule

Example 2) If $f(x)=(2x+5)^3$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule b) with chain rule

Example 3) If $f(x)=(2x+5)^{10}$, find $f'(x)$

Example 4) If $f(x)=\sqrt{2x+5}$, find $f'(x)$

Example 5) Find y' if $y=\frac{1}{4x-3}$

Example 6) Find y' if $y=(3x^2-2x+1)^3$

Differentiation by the Chain Rule - Homework

Find the derivatives of the following:

1. $y = (3x-8)^4$
2. $y = (3x^2+2)^5$
3. $y = 4(x^2+x-1)^{10}$
4. $y = -5(4-9x)^{3/2}$
5. $y = \dfrac{1}{3x-2}$
6. $y = \dfrac{-1}{(x^2-5x-6)^2}$
7. $y = \left(\dfrac{2}{2-x}\right)^2$
8. $y = \dfrac{4x}{(x+1)^2}$
9. $y = \dfrac{-3}{(x^3-x^2+3)^3}$
10. $y = x^3(5x-1)^4$
11. $y = \sqrt{1-t}$
12. $y = \sqrt[3]{3x^3-4x+2}$
13. $y = \dfrac{2}{\sqrt{2x+3}}$
14. $y = \dfrac{-1}{\sqrt{x}+1}$
15. $y = \sqrt{\dfrac{3x}{2x-3}}$

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Differentiation - Chain Rule

Date________________ Period____

Differentiate each function with respect to x.

1) $y = (x^3 + 3)^5$

2) $y = (-3x^5 + 1)^3$

3) $y = (-5x^3 - 3)^3$

4) $y = (5x^2 + 3)^4$

5) $f(x) = \sqrt[4]{-3x^4 - 2}$

6) $f(x) = \sqrt{-2x^2 + 1}$

7) $f(x) = \sqrt[3]{-2x^4 + 5}$

8) $y = (-x^4 - 3)^{-2}$

Worksheet # 13: Chain Rule

1. (a) Carefully state the chain rule using complete sentences.

   (b) Suppose f and g are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = \frac{1}{4}$, and $g'(2) = 2$. Find each of the following:

      i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.

      ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

2. Given the following functions: $f(x) = \sec(x)$, and $g(x) = x^3 - 2x + 1$. Find:

   (a) $f(g(x)) =$

   (b) $f'(x) =$

   (c) $g'(x) =$

   (d) $f'(g(x)) =$

   (e) $(f \circ g)'(x) =$

3. Differentiate each of the following and simplify your answer.

   (a) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$

   (b) $g(t) = \tan(\sin(t))$

   (c) $h(u) = \sec^2(u) + \tan^2(u)$

   (d) $f(x) = xe^{(3x^2+x)}$

   (e) $g(x) = \sin(\sin(\sin(x)))$

4. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2e^{3x}$, $x = 2$

   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$

5. Compute the derivative of $\frac{x}{x^2+1}$ in two ways:

   (a) Using the quotient rule.

   (b) Rewrite the function $\frac{x}{x^2+1} = x(x^2+1)^{-1}$ and use the product and chain rule.

   Check that both answers give the same result.

6. If $h(x) = \sqrt{4 + 3f(x)}$ where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

7. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

   Find: $h'(-1), h'(3)$ and $k'(2)$.

8. Find all x values so that $f(x) = 2\sin(x) + \sin^2(x)$ has a horizontal tangent at x.

9. Comprehension check for derivatives of trigonometric functions:

   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.

   (b) True or False: If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \frac{2}{3}$, then the hypotenuse of the triangle must have length 3.

   (c) Differentiate both sides of the identity $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ to obtain a new trigonometric identity.

Differentiation by the Chain Rule - Classwork

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a) $f(x)=(2x+5)^2$ b) $f(x)=(2x+5)^3$ c. $f(x)=(2x+5)^6$ d) $f(x)=\sqrt{2x+5}$

Yes, multiply first. Yes... Yes but ugh No.

a) causes no problem. b) is also not a problem but multiplying it out so you can take the derivative is a bit of a pain. You are capable of doing c) but clearly do not wish to. But d) can't be done with the knowledge you have.

We now introduce a method of taking derivatives of more complicated expressions. It is called the **chain rule**. If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentable function of x, then $y=f(g(x))$ is a differentable function of x and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ or equivalently, $\frac{d}{dx}[f(g(x))]=f'(g(x))\cdot g'(x)$.

Example 1) If $f(x)=(2x+5)^2$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule

$f(x)=4x^2+20x+25$

$f'(x)=8x+20.$

b) with chain rule

$f'(x)=2(2x+5)\cdot 2=4(2x+5)$

$=8x+20.$

Example 2) If $f(x)=(2x+5)^3$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule

b) with chain rule

$f'(x)=3(2x+5)^2\cdot 2=6(2x+5)^2$

Example 3) If $f(x)=(2x+5)^{10}$, find $f'(x)$

$f'(x)=10(2x+5)^9\cdot 2$

$=20(2x+5)^9$

Example 4) If $f(x)=\sqrt{2x+5}$, find $f'(x)$

$f'(x)=\frac{1}{2}(2x+5)^{-1/2}\cdot 2=\frac{1}{\sqrt{2x+5}}$

Example 5) Find y' if $y=\frac{1}{4x-3}$

$y=(4x-3)^{-1}$

$y'=-(4x-3)^{-2}\cdot 4$

$=-\frac{4}{(4x-3)^2}$

Example 6) Find y' if $y=(3x^2-2x+1)^3$

$y'=3(3x^2-2x+1)^2(6x-2)$

Differentiation by the Chain Rule - Homework

Find the derivatives of the following:

1. $y=(3x-8)^4$

2. $y=(3x^2+2)^5$

3. $y=4(x^2+x-1)^{10}$ ← Power

$y' = 40(x^2+x-1)^9(2x+1)$

Chain ↗

4. $y=-5(4-9x)^{3/2}$

5. $y=\frac{1}{3x-2} = (3x-2)^{-1}$

$y' = -(3x-2)^{-2}\cdot 3$

$= -\frac{3}{(3x-2)^2}$

6. $y=\frac{-1}{(x^2-5x-6)^2}$

7. $y=\left(\frac{2}{2-x}\right)^2 = 4(2-x)^{-2}$

$y' = -8(2-x)^{-3}\cdot -1$

$= \frac{8}{(2x-3)^3}$

8. $y=\frac{4x}{(x+1)^2}$

9. $y=\frac{-3}{(x^3-x^2+3)^3}$

10. $y=x^3(5x-1)^4$

$y' = 3x^2(5x-1)^4 + x^3\cdot 4(5x-1)^3\cdot 5$

Product... → Chain rule!

11. $y=\sqrt{1-t}$

12. $y=\sqrt[3]{3x^3-4x+2}$

$y' = \frac{1}{3}(3x^3-4x+2)^{-2/3}(9x^2-4)$

$= \frac{9x^2-4}{3\sqrt[3]{3x^3-4x+2}^{\,2}}$

13. $y=\frac{2}{\sqrt{2x+3}} = 2(2x+3)^{-1/2}$

$y' = -(2x+3)^{-3/2}\cdot 2$

$= -2(2x+3)^{-3/2}$

14. $y=\frac{-1}{\sqrt{x}+1}$

15. $y=\sqrt{\frac{3x}{2x-3}} = (3x)^{1/2}\cdot(2x-3)^{-1/2}$

Product & Chain ×2

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Differentiate each function with respect to x.

1) $y = (x^3 + 3)^5$

$y' = 5(x^3+3)^4(3x^2)$

2) $y = (-3x^5 + 1)^3$

3) $y = (-5x^3 - 3)^3$

4) $y = (5x^2 + 3)^4$

$y' = 4(5x^2+3)^3(10x)$

5) $f(x) = \sqrt[4]{-3x^4 - 2}$

$f'(x) = \frac{1}{4}(-3x^4-2)^{-3/4}(-12x^3)$

6) $f(x) = \sqrt{-2x^2 + 1}$

7) $f(x) = \sqrt[3]{-2x^4 + 5}$

8) $y = (-x^4 - 3)^{-2}$

$y' = -2(-x^4-3)^{-3}(-4x^3)$