When we examine functions in algebraic form, we can make the following conclusions:

a) all polynomials 1. $\lim_{x \to c} f(x)$ exists 2. $f(c)$ exists 3. $\lim_{x \to c} f(x) = f(c)$ are continuous at all values of x.

b) fractions in the form of $y = \dfrac{f(x)}{g(x)}$ are discontinuous wherever $g(x) = 0$.

c) radicals in the form of $y = \sqrt[\text{odd root}]{f(x)}$ are continuous everywhere.

d) radicals in the form of $y = \sqrt[\text{even root}]{f(x)}$ are discontinuous where $f(x) < 0$.

Examples: Find any points of discontinuity of the following functions.

a) $f(x) = -3x^2 - 5x + 1$ b) $f(x) = \dfrac{x-2}{x^2-4}$ c) $f(x) = \sqrt[3]{x^2 + 2x - 1}$ d) $f(x) = \sqrt{x^2 - x - 6}$

In dealing with continuity of a piecewise function, we need to examine the x-value where the rule changes.

Example 3) $f(x) = \begin{cases} x^2 - 3, x \ge 1 \\ 1 - x, x < 1 \end{cases}$

Example 4) $f(x) = \begin{cases} x^2 + 3x - 2, x \ge -2 \\ -x^2, x < -2 \end{cases}$

Example 5) $f(x) = \begin{cases} 3^{-x} - 1, x \ge -1 \\ \dfrac{1}{x+1}, x < -1 \end{cases}$

Example 6) $f(x) = \begin{cases} \dfrac{x-4}{x^2-16}, x \ne 4 \\ \dfrac{1}{3x-4}, x = 4 \end{cases}$

Find the value of the constant k that will make the function continuous. Verify by calculator.

Example 7) $f(x) = \begin{cases} 3x + 2, x \ge 1 \\ 2k - x, x < 1 \end{cases}$ Example 8) $f(x) = \begin{cases} kx^2, x \ge 2 \\ kx - 6, x < 2 \end{cases}$ Example 9) $f(x) = \begin{cases} k^2 - 12x, x \ge 1 \\ kx, x < 1 \end{cases}$

An important concept in calculus involves the concept of **differentiability**. There are several definitions that you need to know:

Definitions

Differentiability at a point: Function $f(x)$ is differentiable at $x = c$ if and only if $f'(c)$ exists. That is, $f'(c)$ is a real number.

Differentiability on an interval: Function $f(x)$ is differentiable on an interval (a,b) if and only if it is diffferentiable for every value of x on the interval (a,b).

Differentiability: Function $f(x)$ is differentiable if and only if it is differentiable at every value of x in its domain.

The concept of differentiability means, in laymans terms, "smooth." A differentiable curve will have no sharp points in it (cusp points) or places where the tangent line to the curve is vertical. Imagine a train traveling on a set of differentiable tracks and you will never get a derailment. Naturally, if a curve is to be differentiable, it must be defined at every point and its limit must exist everywhere. That implies the following:

Differentiability implies Continuity, Continuity does not imply Differentiability.

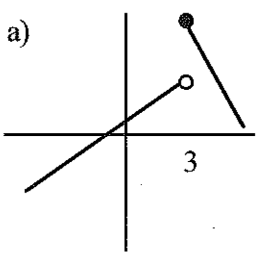
If a function $f(x)$ is differentiable at $x = c$ then it must be continuous also at $x = c$. $\boxed{D \Rightarrow C}$

However, if a function is continuous at $x = c$, it need not be differentiable at $x = c$. $\boxed{\text{Not! } C \Rightarrow D}$

And, if a function is not continuous, then it can't be differentiable at $x = c$. $\boxed{\text{not } C \Rightarrow \text{ not } D}$

Example: determine whether the following functions are continuous, differentiable, neither, or both at the point.

a)



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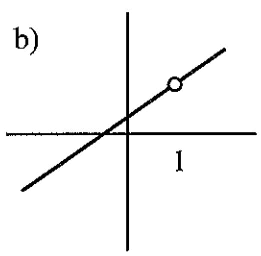

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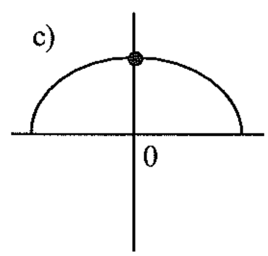

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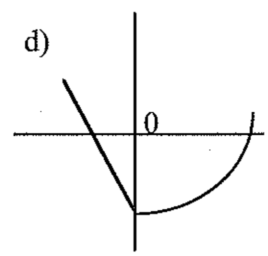

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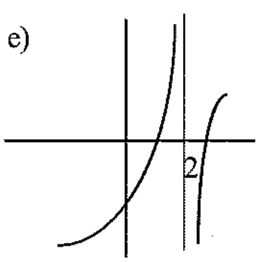

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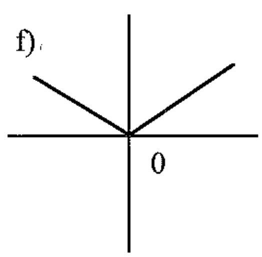

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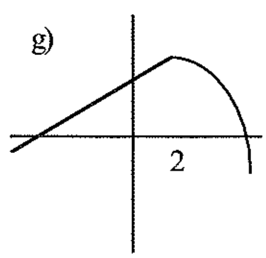

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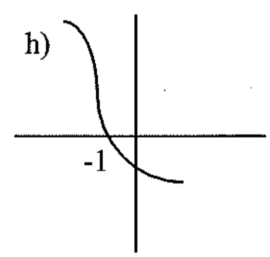


i) $f(x) = x^2 - 6x + 1$

j) $f(x) = \dfrac{x^2 - x - 12}{x + 3}$

k) $f(x) = \sin x$

l) $f(x) = \dfrac{\sin x}{x}$

Example 2) Determine if $f(x)$ is continuous and/or differentiable at the value of the function where the rule changes. Sketch the function.

a) $f(x)=\begin{cases} x^2-6x+10, x\geq 2 \\ 4-x, x<2 \end{cases}$

b) $f(x)=\begin{cases} x^2+x-3, x\geq -1 \\ -x-4, x<-1 \end{cases}$

c) $f(x)=\begin{cases} \sqrt{x+5}, x\geq 4 \\ 4-\sqrt[3]{x-4}, x<4 \end{cases}$

d) $f(x)=\begin{cases} \sin x, x\geq 0 \\ x-3x^2, x<0 \end{cases}$

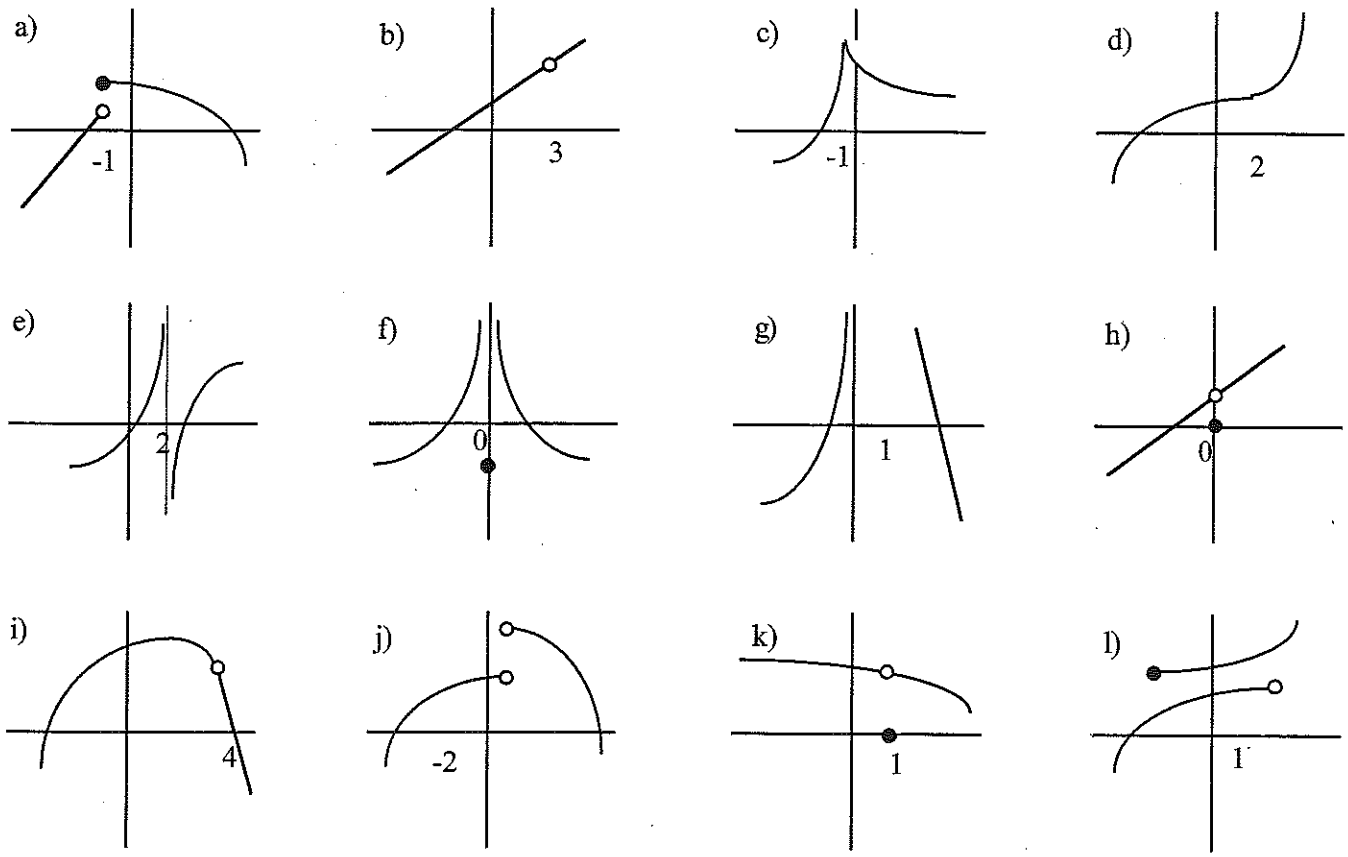
Example 3) Find the values of *a* and *b* that make the function $f(x)$ differentiable.

a) $f(x)=\begin{cases} ax^2+1, x\geq 1 \\ bx-3, x<1 \end{cases}$

b) $f(x)=\begin{cases} ax^3+1, x<2 \\ b(x-3)^2+10, x\geq 2 \end{cases}$

Continuity and Differentiability - Homework

1. In the following graphs determine if the function $f(x)$ is continuous at the marked value of c, and if not, determine for which of the 3 rules of continuity the function fails.



2. Find the value of x where the function is discontinuous.

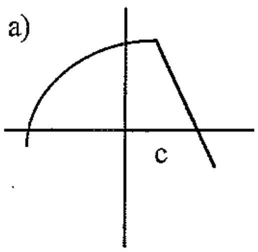
   a. $f(x) = x^3 + 3^x$ b. $f(x) = \dfrac{5}{x^2 - 81}$ c. $f(x) = \dfrac{x^2 + 2x - 24}{x^2 - 36}$ d. $f(x) = \tan x$

3. Find whether the function is continuous at the value where the rule for the function changes.

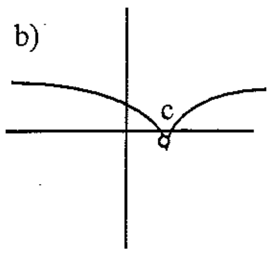
   a. $f(x) = \begin{cases} 8 - x^2, x < 2 \\ 6 - x, x \geq 2 \end{cases}$ b. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 1 + x, x \geq 1 \end{cases}$ c. $f(x) = \begin{cases} 2^x, x < 3 \\ 10 - x, x \geq 3 \end{cases}$

8. For the following, state whether the function is continuous, differentiable, both, or neither at $x = c$

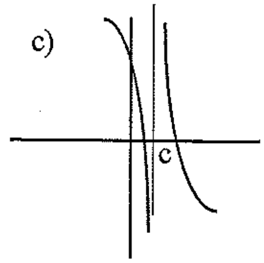
a)



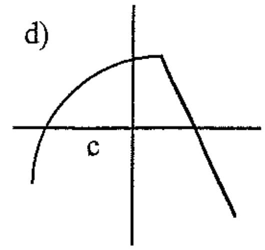
b)



c)



d)



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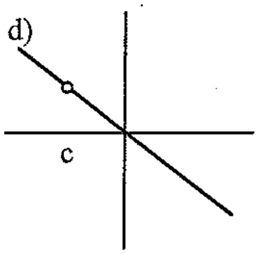

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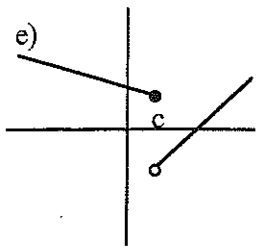

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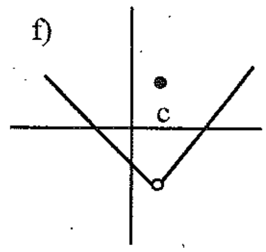

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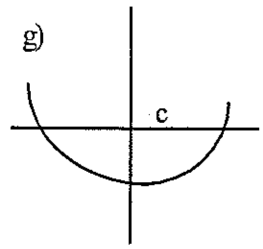

10. For each function, $f(x)$, show work to determine whether the function is continuous or non-continuous, differentiable, or non-differentiable, and sketch the curve. Show work necessary to prove your statements.

a. $f(x) = \begin{cases} x^2, x \geq 0 \\ x, x < 0 \end{cases}$

b. $f(x) = \begin{cases} x^2 + 1, x \geq 0 \\ x^3 + 1, x < 0 \end{cases}$

c. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 2x + 2, x \geq 1 \end{cases}$

d. $f(x) = \begin{cases} x^2 + x - 7, x \geq 2 \\ 5x - 11, x < 2 \end{cases}$

e. $f(x) = \begin{cases} x^4 - 2x^2, x > 1 \\ -1, x \leq 1 \end{cases}$

f. $f(x) = \begin{cases} \sqrt{x} - 3, x > 1 \\ \frac{1}{2}x - \frac{5}{2}, x \leq 1 \end{cases}$